Strongly Cosserat Elastic Lattice and Foam Materials for Enhanced Toughness

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SUMMARY

Some foams exhibit size effects and other phenomena not describable by classical elasticity. These foams are describable by Cosserat elasticity, which is a continuum theory with more freedom than classical elasticity. Cosserat solids have a characteristic length which is greater than zero. Strongly Cosserat elastic materials are considered to be those materials for which the Cosserat characteristic length is substantially greater than the structure size and for which the coupling number is large. Such materials are predicted to exhibit superior toughness. A mechanically isotropic lattice model is presented for the study of foams. Ordinary open cell foams are shown to be weakly Cosserat elastic. If cell rib properties are modified, strongly Cosserat elastic effects can occur in the foam. Anisotropic laminate and fibrous materials can also be made to exhibit strongly Cosserat elastic effects.

1. INTRODUCTION

Toughness is an important property of structural materials. While many foams exhibit a substantial energy absorption to failure in compression as a result of foam rib buckling, the tensile behaviour is typically characterized by abrupt failure[1]. Fibrous composites as well usually lack the ductility found in metals such as mild steel. Improved toughness can be achieved via material microstructure which allows parallel stress paths around holes, cracks and other stress raisers. For example, in synthetic fibrous materials, Awerbuch and Madukhar[2], reported stress concentration factors around small holes which are lower than values predicted from classical elasticity. Such phenomena can be understood in light of the generalized continuum theories which allow additional degrees of freedom associated with the microstructure. Cosserat elasticity[3], in which the points in the continuum can rotate as well as translate, is one of the simplest of these theories. Cosserat elasticity and the related microstructure elasticity theory have a natural characteristic length scale associated with the theory, in contrast with classical elasticity in which there is no such length scale. Generalized continuum theories are therefore of interest in connection with structured materials such as foams and composites, in which the microstructure size is not negligibly small. Cosserat elastic constants can be extracted from experimental measurements of size effects in the torsion and bending rigidity of rods. Several foams have been demonstrated to behave as Cosserat solids by this method[4,5]. Cosserat elasticity predicts stress concentrations[6] around holes to be smaller than expected classically. In this article, generalized continuum theories both for analysis of foams and in the context of guiding the development of new foam material microstructures with superior toughness are considered.

2. ISOTROPIC COSSERAT SOLIDS

In the isotropic Cosserat solid there are six elastic constants, in contrast to the classical elastic solid in which there are two. Several combinations of Cosserat elastic constants have dimensions of length and are referred to as characteristic lengths. The constitutive equations for a linear isotropic Cosserat elastic solid [7] also known as a micropolar solid [8] are:

$$\sigma_{kl} = \lambda \varepsilon_{rr}\, \delta_{kl} + (2\mu + \kappa)\varepsilon_{kl} + \kappa \varepsilon_{klm}(r_m - \phi_m) \tag{1}$$

$$m_{kl} = \alpha\, \phi_{r,r}\, \delta_{kl} + \beta \phi_{k,l} + \gamma \phi_{l,k} \tag{2}$$

The usual summation convention for repeated indices is used throughout and the comma denotes differentiation with respect to spatial coordinates. σ_{kl} is the force stress, which is a symmetric tensor in classical elasticity but it is asymmetric here. m_{kl} is the couple stress or moment per unit area, $\varepsilon_{kl} = (u_{k,l} + u_{l,k})/2$ is the small strain, u_k is the displacement, and ε_{klm} is the permutation symbol. The micro-rotation ϕ_k in Cosserat elasticity is kinematically distinct from the macro-rotation $r_k = (e_{klm} u_{m,l})/2$. In three dimensions, the isotropic Cosserat elastic solid requires six elastic constants λ, μ, α, β, γ, and κ for its description. The following technical constants derived from them are beneficial in terms of physical insight. These were discussed by Eringen[8] and Gauthier and Jahsman[9]:

Young's modulus E	=	$(2\mu+\kappa)(3\lambda+2\mu+\kappa)/(2\lambda+2\mu+\kappa)$
Shear modulus G	=	$(2\mu+\kappa)/2$
Poisson's ratio ν	=	$\lambda/(2\lambda+2\mu+\kappa)$
Characteristic length for torsion l_t	=	$[(\beta+\gamma)/(2\mu+\kappa)]^{1/2}$
Characteristic length for bending l_b	=	$[\gamma/2(2\mu+\kappa)]^{1/2}$
Coupling number N	=	$[\kappa/2(\mu+\kappa)]^{1/2}$ (dimensionless) and
Polar ratio Ψ	=	$(\beta+\gamma)/(\alpha+\beta+\gamma)$ (dimensionless)

The Cosserat characteristic lengths govern the size scale at which deviations from classical elasticity are observed. For example, size effects occur in the bending and torsion of rods[9]; the rigidity of such rods becomes significantly larger than that of a classical rod when the rod diameter is about ten times the characteristic length. The coupling number governs the magnitude of the Cosserat elastic effects[9].

The characteristic lengths of cellular and composite materials can be obtained from microstructural analysis, in terms of the material properties of the constituents. The Cosserat characteristic lengths are predicted to be of the order of the size of the structural elements for several structures such as two dimensional lattices[10]. In some materials such as particle reinforced composites, the characteristic lengths are predicted to be zero[11] and have been experimentally demonstrated to be zero[4,9] in such materials. Experimental work on foams discloses characteristic lengths somewhat larger than the foam cells, as recently reviewed[5].

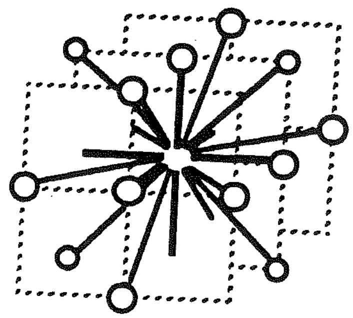
We define strongly Cosserat elastic materials to be those materials for which the Cosserat characteristic lengths are substantially greater than the structure size and for which the coupling number N is large (its range is from zero to unity). In the 'usual' case in which l is small, comparable to the size of structural elements in a foam or composite, the effect of the Cosserat elasticity will manifest itself in the distribution of stress near crack tips[12] and in the reduction of stress concentration around small holes[6]. The effect in this case will be to increase the toughness[12]. In strongly Cosserat elastic materials, dramatic increases in toughness are anticipated, and reduction of stress concentrations associated with holes and other inhomogeneities much larger than the structure size.

3. LATTICES AS STRONGLY COSSERAT ELASTIC MATERIALS

3.1 Mechanically isotropic lattices

We consider the lattice model of Tauchert[13] with the aim of determining the attainable strength of Cosserat elastic effects, given modifications of the rib structure. If we choose curved ribs, they deform principally by bending or twisting, and the model approximates the behaviour of an open cell foam. In this model, shown in Figure 1, mass points in a structurally cubic lattice are connected by ribs (bars) capable of supporting axial, shear, flexural, and torsion loads. Each lattice point can rotate as well as translate, corresponding to the kinematical freedom of the points in the Cosserat continuum representation.

Figure 1 Lattice structure with ribs connecting nearest neighbour points and next nearest neighbour points. Ribs can be straight or cured



A feature of the lattice model which is of interest from the perspective taken here is that the bars are not assumed to have a special shape, so that the stiffness coefficients are independent. They are:

$$a_{ij} = E_{ij}A_{ij}/LL_{ij}$$
$$b_{ij} = 12E_{ij}I_{ij}/LL_{ij}^{3}$$
$$c_{ij} = G_{ij}J_{ij}/LL_{ij}, \text{ and}$$
$$d_{ij} = 4E_{ij}/LL_{ij}$$

in which E_{ij} is Young's modulus
G_{ij} is the shear modulus
A_{ij} is the cross sectional area
I_{ij} is the moment of inertia
J_{ij} is the polar moment of inertia of a bar denoted by indices (ij),
and L is the lattice spacing.

Bars with i=j connect nearest neighbours along the orthogonal cubic symmetry axes, and those with i≠j connect next nearest neighbours.

The lattice is structurally cubic, but mechanically other symmetries such as orthotropy or isotropy can be achieved by choice of the properties of the ribs. We consider the isotropic case since it is the simplest. Many solutions of boundary value problems as well as experimental results are available for the isotropic solids; it is possible to experimentally determine[4,5] all six elastic constants for the isotropic Cosserat solid. To calculate the Cosserat elastic constants in the isotropic lattice, we compare the orthotropic equations of motion and constitutive relations of Tauchert[13] in terms of structural parameters term by term with the isotropic continuum constitutive equations of Eringen[8]. For isotropy we must have:

$$a_{12}=a_{21}=a_{13}=a_{31}=a_{23}=a_{32}$$

and $$a_{11}=a_{22}=a_{33}$$

and similarly for the b, c, and d coefficients. Additional conditions for isotropy obtained by comparing the structural and continuum equations are that:

$$a_{11}=a_{12}+b_{22}-b_{12}$$
$$c_{11}=C_{12}+d_{11}/4,\ a_{11}=a_{12}$$
$$b_{11}=b_{12}$$

For the mechanically isotropic lattice, a hydrostatic deformation gives rise to rib axial deformation without bending or torsion. The elastic constants for isotropic material are obtained below from the anisotropic relations of Tauchert [13] in terms of the rib extension coefficients a, rib bend coefficients b and d, and rib twist coefficients c,

Shear modulus:

$$G = a_{12} + b_{11}/2 + b_{12} \qquad (3)$$

Poisson's ratio:

$$\nu = (a_{12}-b_{12})/[4a_{12}+b_{11}] \qquad (4)$$

Coupling number:

$$N = [(b_{11}+4b_{12})/(2a_{12}+2b_{11}+6b_{12})]^{1/2} \qquad (5)$$

Characteristic length, bending:

$$l_b = [(c_{12}+d_{11}/4+3d_{12}/4)/(4a_{12}+2b+4b_{12})]^{1/2} \qquad (6)$$

Characteristic length, torsion:

$$lt = [(2c_{12}+d_{11}/4+d_{12}/2)/(a_{12}+b_{11}/2+b_{12})]^{1/2} \qquad (7)$$

Polar ratio:

$$\Psi = (2c_{12}+d_{11}/4+d_{12}/2)/(3c_{12}+d_{11}/4+d_{12}) \qquad (8)$$

For lattices containing slender straight ribs of diameter d, $b/a = 3d^2/4L^2$; the assumed slenderness implies $b/a \ll 1$. The classical shear modulus is then governed mostly by the rib stiffness in extension, unlike most open cell foams. Moreover, $\nu \approx 0.25$. The Cosserat characteristic lengths are much smaller than the lattice spacing and N « 1; such a lattice is nearly classical. For ribs which are solid thick bars of length three times the diameter (if they were any thicker, the structure would no longer be a lattice) we obtain $\nu = 0.24$, $N = 0.4$, $l_b = 0.08L$, $l_t = 0.15L$, and $\Psi = 0.6$ (the last equality is true for bars made of any solid straight ribs). This is still weakly Cosserat elastic: the characteristic lengths are smaller than the structure size, and the coupling number is not very large.

We may envisage a lattice of curved ribs; as discussed below, this model is representative of open cell foams. We compare the extensional stiffness of a straight rod of circular cross section of diameter d with that of a rod cured into a half circle of radius R (14) and find the ratio of the extensional rigidities to be $(d/R)^2/_8$, so that with $L = 2R$, we have $b/a = 1.5$. For this case, $\nu = -0.18$, $N = 0.7$, ,$l_b = 0.21L$, and $l_t = 0.38L$. The results are essentially independent of rib aspect ratio. The Cosserat effects are now larger but the characteristic lengths are still smaller than the structure size. Even if we could achieve b » a, the ,l's would still be less than the lattice spacing L.

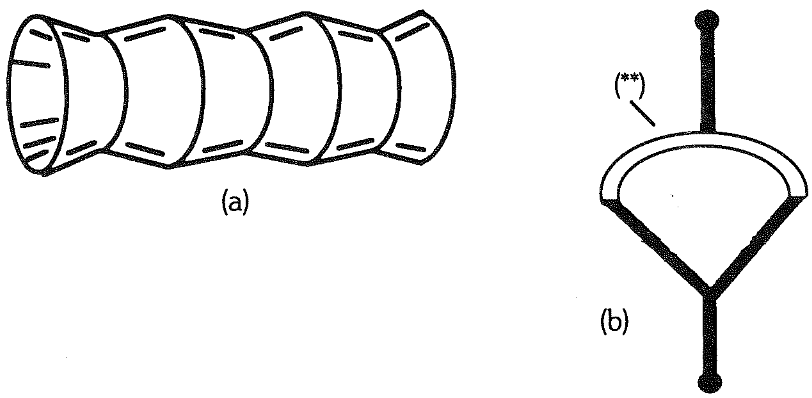
3.2 Ways to achieve strong Cosserat effects

The characteristic lengths can be made much larger than the lattice size if the rib torsion coefficients c are made large compared with the rib extensional coefficients a and the bend coefficients b and d as seen in Equation 6 and 7. Structures of this type give rise to large characteristic lengths, but increasing the rib torsion rigidity c alone will not allow $N \approx 1$, as seen in Equation 5. The unique role of rib torsion is at first surprising since the spatial average of bending and twisting moments upon ribs represent components of the couple stress. However, rib twisting differs structurally from rib bending in that a uniform axial or shear (but not hydrostatic) deformation of the lattice causes rib extension and bending but not twisting. The bending couples are of alternating sign so the couple stress vanishes for uniform deformation, in agreement with the continuum view[9]. The modulus for uniform deformation is relevant here since it appears in the denominator in the definition of the characteristic lengths.

Ribs which are much more rigid in torsion than in extension or bending can be made in the form of corrugated tubules, as shown in Figure 2a. Alternatively, the ribs can be made of negative Poisson's ratio material [15], so that the rib shear modulus can substantially exceed the Young's modulus. Another possibility for the achievement of large characteristic lengths is to make the rib axial stiffness negative, a possibility which was recognized by Berglund[11]. Such a rib would be unstable in isolation, but

Figure 2 Cell ribs which give rise to strongly Cosserat elastic foams.
(a) Cell rib as a corrugated tubule is stiff in torsion but compliant in compression and bending
(b) Cell rib containing a tied arch; flexible portion is shown asindicated(**). The configuration shown is stable. Under sufficient compression the arch exhibits another equilibrium position which is unstable and which corresponds to negative compressional stiffness.

A macroscopically homogeneous rib with a negative Poisson's ratio also will give rise to strongly Cosserat elastic foams



as part of a lattice, stability would follow from satisfaction of the continuum bounds (constitutive inequalities)[8]:

$$G \geq 0$$
$$-1 \leq \nu \leq 0.5$$
$$l_b \geq 0. \; l_t \geq 0$$
$$0 \leq N \leq 1$$
$$0 \leq \Psi \leq 1.5$$

These bounds were derived from the requirement of positive definiteness of the strain energy as in classical elasticity. Ribs of negative axial stiffness can be made of a subcell containing spring elements with prestrain[16]. Another possibility is a cell rib containing a tied arch, shown in Figure 2b. The flexible portion is shown as indicated(**). The configuration shown is stable. Under sufficient compression the arch exhibits another equilibrium position which is unstable and which corresponds to negative compressional stiffness[17]. The cell ribs still should have a positive bend stiffness, so the rib itself should be composite in nature, e.g. we may envisage a segment of negative stiffness surrounded by structure of positive stiffness offset from the neutral axis of bending.

4. COMPARISON OF LATTICES WITH FOAMS

The mechanics of open cell foam materials is governed by the bending of the ribs[1] in contrast to the above lattices of straight ribs in which the rib extensional stiffness dominates the stiffness. As for the classical elastic moduli, the above lattice model assuming curved ribs gives:

$$G_{foam}/G_{solid} \propto (\rho_{foam}/\rho_{solid})^2$$

in which ρ refers to density and solid refers to the solid from which the foam ribs is made. This is in agreement with experiment and with elementary theory for open cell foams[1] and in contrast to the behaviour of a lattice of straight ribs in which the density ratio is raised to the first power rather than the second:

$$G_{foam}/G_{solid} \propto (\rho_{foam}/\rho_{solid})$$

The Poisson's ratio of conventional foams is about +0.3 and for a lattice of straight ribs it is +0.25. The structure of conventional foam cells is tetrakaidecahedral, which differs considerably from the idealized cubical cells of the lattices. In negative Poisson's ratio polymeric foams[15], ν is as small as -0.7, compared with -0.18 in lattices of curved semicircular ribs. The effect of rib curvature in the lattice model is to reduce the Poisson's ratio. The actual ribs in re-entrant foam can be convoluted, with more curvature than the semicircle model.

As for Cosserat elasticity in foams, the experimental characteristic lengths exceed the foam cell size in conventional foams of various density, mostly closed cell[4,5] and in open cell negative Poisson's ratio foams[5]. The lattice model, which is open cell, does not account for these observations, even if curved ribs are allowed. In dense closed cell foam, $l_t > l_b$ was observed experimentally; in low density closed cell foam, $l_t < l_b$; in the open cell lattice model, $l_t > l_b$. Simple foam models[1] consider rib bending to predict classical elastic behaviour; they do not deal with rib twist or Cosserat elasticity. As for closed cell conventional foams, the lattice model does not include any effect of the membrane or plate elements contained within closed cell foams, so we cannot make a direct comparison. Nevertheless the lattice model is suggestive in terms of the role of curved ribs. Foams of conventional structure, although they exhibit $l > d_{cell}$ are experimentally not strongly Cosserat elastic solids since the characteristic lengths are not dramatically larger than the cell size, and N is in the range 0.2 to 0.3, well below its upper bound.

The negative Poisson's ratio foams[15] differ structurally from the conventional ones in that the cell ribs of the former are sharply curved, even convoluted. The curved and convoluted ribs in the negative Poisson's ratio foam structure suggests, in the context of the lattice model, an intensification of the Cosserat elastic effects; however the lattice model does not incorporate the 're-entrant' aspect of the cell structure. Further experimental characterization of these materials is in order.

We cannot exclude the possibility that more general continuum theories, such as that of Mindlin[18] may apply to cellular materials. The Mindlin microstructure theory makes use of 18 elastic constants in the isotropic case. It is difficult to apply this theory to experimental results since the relevant boundary value problems have not been solved. Non-local elasticity[19,20] is another possibility, however, again the boundary value problems for torsion and bending have not been solved.

5. ANISOTROPIC STRONGLY COSSERAT ELASTIC MATERIALS

Laminates and fibrous materials are considered here for comparison with foams. We make use of the anisotropic generalized continuum analysis of laminates[21], and we proceed in a way similar to the above with the aim of achieving strongly Cosserat elastic effects. The laminate, with some change in notation from ref. (21), has stiff layers of thickness h, shear modulus G_f, Poisson's ratio ν_f and volume fraction V_f, and has compliant layers of shear modulus G_m and volume fraction $1\text{-}V_f$. The Cosserat characteristic length is given by

$$l = \sqrt{3}h[V_f G_f / 6G_m(1-\nu_f)]^{1/2} \tag{9}$$

The original results [21] were given in terms of 'couple stress theory' which is a special case of Cosserat elasticity corresponding to N = 1. The characteristic lengths are defined somewhat differently in these theories, so that the Cosserat length for torsion is √3 times the couple stress length.

There are two principal stiffnesses in this anisotropic material, corresponding to orthogonal direction of applied load

$$G_{Voigt} = G_f V_f + G_m(1 - V_f) \tag{10}$$

$$G_{Reuss} = [V_f/G_f + V_m/G_m]^{-1} = [V_f + (G_f/G_m)(1 - V_f)]^{-1} \tag{11}$$

The Cosserat κ is given by $\kappa = G_f V_f$, so there are two coupling numbers

$$N_{Voigt} = [G_f V_f/(2G_{voigt} + G_f V_f)]^{1/2} \quad (12)$$

$$N_{Reuss} = [G_f V_f/(2G_{reuss} + G_f v_f)]^{1/2} \quad (13)$$

To achieve strongly Cosserat elastic effects in the laminate, let $G_f/G_m \rightarrow \propto$ and $V_f \rightarrow 1$ in Equation 9 which gives $l/h \rightarrow \propto$ so that the characteristic length becomes much larger than the structure size. $V_f \rightarrow 1$ in Equation 10 gives a Voigt stiffness approaching that of the stiff layers so that $N_{voigt} \rightarrow 0\text{-}577$ which is substantial but still less than the upper bound of 1. The laminate can be made as stiff in the Reuss direction as the stiff layers as well by requiring $(G_f/G_m)(l - V_f) \rightarrow 0$ in Equation 11, which can be achieved simultaneously with the above limiting procedure. Then $N_{Reuss} \rightarrow 0\ 577$. Such a laminate is regarded as strongly Cosserat elastic. The coupling number for the Reuss direction can be made to approach 1 only if stiffness in that direction is sacrificed as seen in Equation 11 and 13. A drawback of the laminate structure is that if tensile stress is applied perpendicular to the laminae, all of the stress passes through the soft matrix phase. Even if this phase is made very thin to maintain the stiffness of the laminate, the laminate strength for this direction of loading is likely to be low. Strength could be improved by using a dovetail structure, but that would complicate the analysis.

The theory of fibrous composites by Hlavacek[22] incorporates the freedom of a Mindlin type[18] generalized continuum, which includes the Cosserat and classical continua as special cases. We consider here the Cosserat elastic constants only. Analysis of the results of this theory discloses the torsional characteristic length l can be made arbitrarily large compared with the fibres if the fibres are much stiffer than the matrix. If, in addition we allow the matrix volume fraction to tend to zero, the results for torsion about the fibre axis are $G_{composite}/G_m \rightarrow 4.$ and $N \rightarrow 0.577$. The stiffness result is considered unphysical since we would expect $G_{composite} \rightarrow G_f$ as the fraction of matrix is reduced to zero. However since the model assumes circular fibres for which the volume fraction can never approach unity, difficulties in limiting procedures are not surprising.

To summarize, if the matrix material between laminae (or between fibres) is made very soft and the matrix layers made very thin, a characteristic length can be made much larger than the structural elements, resulting in strongly Cosserat elastic anisotropic materials. In the case of laminates it is possible to achieve this with little sacrifice in stiffness.

6. DISCUSSION

The lattice model is of interest in part because it can be made structurally isotropic, which facilitates comparison with experiments on foams and aids in understanding the mechanics. Lattice type models have also been considered[11,23], in which the authors pointed out that the Cosserat characteristic length cannot substantially exceed the structure size unless the torsion rigidity of a rib can be decoupled from the axial rigidity of the same rib, a situation regarded as impossible. The present study offers just that kind of decoupling which permits strongly Cosserat elastic solids. Physical realization of such a solid is not straightforward, since the foam must either be made from unusual materials or with an unusual rib structure. Physically, Cosserat elastic effects arise as a result of bending or twisting of structural elements within a material. Such effects certainly occur in foams. In foams there is, however a competing effect in which cells at a free surface are incomplete and therefore carry little load[24]. This surface effect reduces the rigidity of thin specimens of foam[24], leading to the opposite kind of size effect as that seen in Cosserat solids. Which effect predominates will depend on details of the cell structure; the issue is as yet not well understood.

In the lattice or foam type strongly Cosserat elastic solids considered here, there is a tradeoff between the Cosserat constants l and N and the classical stiffness E: large l and N is associated with a reduced value of E. This situation is reminiscent of the difficulty of simultaneously achieving high stiffness, strength and toughness in known materials. This tradeoff might be escaped via non-linearly Cosserat elastic materials in which the Cosserat type effects do not manifest themselves until a critical stress is reached. Alternatively, attention could be directed to applications which require a compliant but tough foam.

As for laminates and fibrous solids, it is easy to achieve strongly Cosserat elastic effects, however the interpretation of the elastic constants and of the effect upon the behaviour of the material is not as straightforward as in the isotropic case. In a structural view, laminated or fibrous microstructures with very compliant matrix materials are also seen to be beneficial in achieving high toughness, as articulated by Cook and Gordon[25] in the context of arresting propagating cracks. A structural perspective was taken by Chiang[26] who analytically demonstrated that for typical fibrous composites, the use of homogeneous classical elasticity in calculating stress intensity factors for cracks is warranted only if the crack size is at least three orders of magnitude larger than the fibre diameter. In the Cosserat view, improved toughness results from amelioration of stress concentration factors around holes and flaws, so that crack initiation is prevented.

As for practical, tough, strongly Cosserat elastic foam materials, we have considered the possibility of foams in which the ribs themselves have a negative Poisson's ratio. One way this could be done is to create a hierarchical foam in which the largest cells have the typical convex tetrakaidecahedral shape, and the ribs are themselves cellular, with a re-entrant structure[15]. An alternative is to make the foam from a solid which achieves a negative Poisson's ratio on a smaller cellular scale[27] or via micro-lamination[28].

7. CONCLUSION

1. Open cell foams are predicted to be weakly Cosserat elastic, even if the cell ribs are substantially curved.
2. Isotropic strongly Cosserat elastic materials can be made from lattices or open cell foams which have ribs which (i) are corrugated tubules, (ii) have a negative Poisson's ratio, or (iii) contain elements of negative stiffness.
3. Anisotropic strongly Cosserat elastic materials can be made in fibrous or laminated geometries provided the ratio between stiffnesses of the constituents is large.

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